OPERATOR INEQUALITIES OF JENSEN TYPE

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ABSTRACT. We present some generalized Jensen type operator inequalities involving sequences of self-adjoint operators. Among other things, we prove that if $f : [0, \infty) \to \mathbb{R}$ is a continuous convex function with $f(0) \leq 0$, then

$$\sum_{i=1}^{n} f(C_i) \leq f\left(\sum_{i=1}^{n} C_i\right) - \delta_f \sum_{i=1}^{n} \widetilde{C}_i \leq f\left(\sum_{i=1}^{n} C_i\right)$$

for all operators C_i such that $0 \leq C_i \leq M \leq \sum_{i=1}^{n} C_i \ \ (i = 1, \ldots, n)$ for some scalar $M \geq 0$, where $\widetilde{C_i} = \frac{1}{2} - \left|\frac{C_i}{M} - \frac{1}{2}\right|$ and $\delta_f = f(0) + f(M) - 2f\left(\frac{M}{2}\right)$.

1. Introduction and Preliminaries

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and I denote the identity operator. If $\dim \mathscr{H} = n$, then we identify $\mathbb{B}(\mathscr{H})$ with the C^*-algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with complex entries. Let us endow the real space $\mathbb{B}_h(\mathscr{H})$ of all self-adjoint operators in $\mathbb{B}(\mathscr{H})$ with the usual operator order $\leq$ defined by the cone of positive operators of $\mathbb{B}(\mathscr{H})$.

If $T \in \mathbb{B}_h(\mathscr{H})$, then $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ and $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$ are called the bounds of T. We denote by $\sigma(J)$ the set of all self-adjoint operators on $\mathscr{H}$ with spectra contained in J. All real-valued functions are assumed to be continuous in this paper. A real valued function f defined on an interval J is said to be operator convex if $f(\lambda A+(1-\lambda)B) \leq \lambda f(A)+(1-\lambda)f(B)$ for all $A, B \in \sigma(J)$ and all $\lambda \in [0, 1]$. If the function f is operator convex, then the so-called Jensen operator inequality $f(\Phi(A)) \leq \Phi(f(A))$ holds for any unital positive linear map Φ on $\mathbb{B}(\mathscr{H})$ and any $A \in \sigma(J)$. The reader is referred to [3, 4, 8] for more information about operator convex functions and other versions of the Jensen operator inequality. It should be remarked that if f is a real convex function, but not operator convex, then the Jensen operator inequality may not hold. To see this, consider the convex (but not operator convex) function $f(t) = t^4$ defined on $[0, \infty)$ and the positive mapping $\Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ defined by

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$\Phi((a_{ij})_{1\le i,j\le 3}) = (a_{ij})_{1\le i,j\le 2}$ for any $A = (a_{ij})_{1\le i,j\le 3} \in \mathcal{M}_3(\mathbb{C})$. If

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix},$$

then there is no relationship between

$$f(\Phi(A)) = \begin{pmatrix} 36 & 46 \\ 46 & 59 \end{pmatrix} \quad \text{and} \quad \Phi(f(A)) = \begin{pmatrix} 36 & 48 \\ 48 & 68 \end{pmatrix}$$

in the usual operator order.

Recently, in [6] a version of the Jensen operator inequality was given without operator convexity as follows:

Theorem A. [6, Theorem 1] Let $(A_1, \ldots, A_n)$ be an n-tuple of operators $A_i \in \mathbb{B}_h(\mathscr{H})$ with bounds m_i and M_i, $m_i \le M_i$, and let $(\Phi_1, \ldots, \Phi_n)$ be an n-tuple of positive linear mappings Φ_i on $\mathbb{B}(\mathscr{H})$ such that $\sum_{i=1}^n \Phi_i(I) = I$. If

$$(m_C, M_C) \cap [m_i, M_i] = \emptyset \tag{1.1}$$

for all $1 \le i \le n$, where m_C and M_C with $m_C \le M_C$ are bounds of the self-adjoint operator $C = \sum_{i=1}^n \Phi_i(A_i)$, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{1.2}$$

holds for every convex function $f : J \to \mathbb{R}$ provided that the interval J contains all m_i, M_i; see also [7].

Another variant of the Jesnen operator inequality is the so-called Jensen–Mercer operator inequality [5] asserting that if f is a real convex function on an interval $[m, M]$, then

$$f\left(M + m - \sum_{i=1}^n \Phi_i(A_i)\right) \le f(M) + f(m) - \sum_{i=1}^n \Phi_i(f(A_i)),$$

where $\Phi_1, \cdots, \Phi_n$ are positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$ and $A_1, \cdots, A_n \in \sigma([m, M])$.

Recently, in [9] an extension of the Jensen–Mercer operator inequality was presented as follows:

Theorem B.[9, Corollary 2.3] Let f be a convex function on an interval J. Let $A_i, B_i, C_i, D_i \in \sigma(J)$ $(i = 1, \cdots, n)$ such that $A_i + D_i = B_i + C_i$ and $A_i \le m \le B_i, C_i \le M \le D_i$. Let $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with

$\sum_{i=1}^n \Phi_i(I) = I$. Then

$$f\left(\sum_{i=1}^n \Phi_i(B_i)\right) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)) + \sum_{i=1}^n \Phi_i(f(D_i)). \tag{1.3}$$

The authors of [9] used inequality (1.3) to obtain some operator inequalities. In particular, they gave a generalization of the Petrović operator inequality as follows:

Theorem C.[9, Corollary 2.5] Let $A, D, B_i \in \sigma(J)$ $(i = 1, \cdots, n)$ such that $A + D = \sum_{i=1}^n B_i$ and $A \leq m \leq B_i \leq M \leq D$ $(i = 1, \cdots, n)$ for two real numbers $m < M$. If f is convex on J, then

$$\sum_{i=1}^n f(B_i) \leq (n-1) f\left(\frac{1}{n-1} A\right) + f(D).$$

If $f : [0, \infty) \to \mathbb{R}$ is a convex function such that $f(0) = 0$, then

$$f(a) + f(b) \leq f(a + b) \tag{1.4}$$

for all scalars $a, b \geq 0$. However, if the scalars a, b are replaced by two positive operators, this inequality may not hold. For example if $f(t) = t^2$ and A, B are the following two positive matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

then a straightforward computation reveals that there is no relationship between $A^2 + B^2$ and $(A + B)^2$ under the operator order. Many authors tried to obtain some operator extensions of (1.4). In [10], it was shown that

$$f(A + B) \leq f(A) + f(B)$$

for all non-negative operator monotone functions $f : [0, \infty) \to [0, \infty)$ if and only if $AB + BA$ is positive.

Another operator extension of (1.4) was established in [9]

Theorem D. [9, Corollary 2.9] If $f : [0, \infty) \to [0, \infty)$ is a convex function with $f(0) \leq 0$, then $f(A) + f(B) \leq f(A + B)$ for all invertible positive operators A, B such that $A \leq MI \leq A + B$ and $B \leq MI \leq A + B$ for some scalar $M \geq 0$.

Some other operator extensions of (1.4) can be found in [1, 2, 11]. In this paper, as a continuation of [9], we extend inequality (1.3), refine (1.3) and improve some

of our results in [9]. Some applications such as further refinements of the Petrović operator inequality and the Jensen–Mercer operator inequality are presented as well.

2. Results

To presenting our results, we introduce the abbreviation:

$$\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)$$

for $f : [m, M] \to \mathbb{R}$, $m < M$.

We need the following lemma may be found in [7, Lemma 2]. We give a proof for the sake of completeness.

Lemma 2.1. *Let $A \in \sigma([m, M])$, for some scalars $m < M$. Then*

$$f(A) \le \frac{M-A}{M-m} f(m) + \frac{A-m}{M-m} f(M) - \delta_f \widetilde{A} \tag{2.1}$$

holds for every convex function $f : [m, M] \to \mathbb{R}$, where

$$\widetilde{A} = \frac{1}{2} - \frac{1}{M-m}\left|A - \frac{m+M}{2}\right|.$$

If f is concave on $[m, M]$, then inequality (2.1) is reversed.

Proof. First assume that $a, b \in [m, M]$ and $\lambda \in [0, 1/2]$ so that $\lambda \le 1 - \lambda$. Then

$$\begin{aligned} f(\lambda a + (1-\lambda)b) &= f\left(2\lambda \frac{a+b}{2} + (1-2\lambda)b\right) \\ &\le 2\lambda f\left(\frac{a+b}{2}\right) + (1-2\lambda) f(b) \\ &= \lambda f(a) + (1-\lambda) f(b) - \lambda\left(f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right). \end{aligned}$$

It follows that

$$\begin{aligned} &f(\lambda a + (1-\lambda)b) \\ &\quad \le \lambda f(a) + (1-\lambda) f(b) - \min\{\lambda, 1-\lambda\}\left(f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right) \end{aligned} \tag{2.2}$$

for all $a, b \in [m, M]$ and all $\lambda \in [0,1]$. If $t \in [m, M]$, then by using (2.2) with $\lambda = \frac{M-t}{M-m}$, $a = m$ and $b = M$ we obtain

$$\begin{aligned} f(t) = f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) &\leq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\ &- \min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\}\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right) \end{aligned} \tag{2.3}$$

for any $t \in [m, M]$. Since $\min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|$, we have from (2.3) that

$$\begin{aligned} f(t) &\leq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\ &- \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right), \end{aligned} \tag{2.4}$$

for all $t \in [m, M]$. Now if $A \in \sigma([m, M])$, then by utilizing the functional calculus to (2.4) we obtain (2.1). □

In the next theorem we present a generalization of [9, Theorem 2.1].

Theorem 2.2. *Let $\Phi_i, \overline{\Phi}_i, \Psi_i, \overline{\Psi}_i$ be positive linear mappings on $\mathbb{B}(\mathscr{H})$ such that $\sum_{i=1}^{n_1} \Phi_i(I) = \alpha I$, $\sum_{i=1}^{n_2} \overline{\Phi}_i(I) = \beta I$, $\sum_{i=1}^{n_3} \Psi_i(I) = \gamma I$, $\sum_{i=1}^{n_4} \overline{\Psi}_i(I) = \delta I$ for some real numbers $\alpha, \beta, \gamma, \delta > 0$. Let A_i $(i = 1, \ldots, n_1)$, D_i $(i = 1, \ldots, n_2)$, C_i $(i = 1, \ldots, n_3)$ and B_i $(i = 1, \ldots, n_4)$ be operators in $\sigma(J)$ such that $A_i \leq m \leq B_i, C_i \leq M \leq D_i$ for two real numbers $m < M$. If*

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) + \frac{1}{\delta}\sum_{i=1}^{n_2} \overline{\Phi}_i(D_i) = \frac{1}{\gamma}\sum_{i=1}^{n_3} \Psi_i(C_i) + \frac{1}{\beta}\sum_{i=1}^{n_4} \overline{\Psi}_i(B_i), \tag{2.5}$$

then

$$\begin{aligned} f\left(\frac{1}{\gamma}\sum_{i=1}^{n_3} \Psi_i(C_i)\right) + f\left(\frac{1}{\beta}\sum_{i=1}^{n_4} \overline{\Psi}_i(B_i)\right) &\leq \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i\left(f(A_i)\right) + \frac{1}{\delta}\sum_{i=1}^{n_2} \overline{\Phi}_i\left(f(D_i)\right) - \delta_f \widetilde{X} \\ &\leq \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i\left(f(A_i)\right) + \frac{1}{\delta}\sum_{i=1}^{n_2} \overline{\Phi}_i\left(f(D_i)\right) \end{aligned} \tag{2.6}$$

holds for every convex function $f : J \to \mathbb{R}$, where

$$\widetilde{X} = 1 - \frac{1}{M-m}\left(\left|\frac{1}{\beta}\sum_{i=1}^{n_4} \overline{\Psi}_i(B_i) - \frac{m+M}{2}\right| + \left|\frac{1}{\gamma}\sum_{i=1}^{n_3} \Psi_i(C_i) - \frac{m+M}{2}\right|\right).$$

If f is concave, then the reverse inequalities are valid in (2.6).

Proof. We prove only the case when f is convex. Let $[m, M] \subseteq J$. It follows from the convexity of f on J that

$$f(t) \geq \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) \tag{2.7}$$

for all $t \in J \setminus [m, M]$. Hence, by $A_i \leq m$ and $D_i \geq M$ we have

$$f(A_i) \geq \frac{M-A_i}{M-m} f(m) + \frac{A_i-m}{M-m} f(M) \quad (i = 1, \cdots, n_1) \tag{2.8}$$

and similarly

$$f(D_i) \geq \frac{M-D_i}{M-m} f(m) + \frac{D_i-m}{M-m} f(M) \quad (i = 1, \cdots, n_2). \tag{2.9}$$

Applying the positive linear mappings Φ_i and $\overline{\Phi}_i$, respectively, to both sides of (2.8) and (2.9) and summing we get

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \geq \frac{M - \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)}{M-m} f(m) + \frac{\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) - m}{M-m} f(M) \tag{2.10}$$

and

$$\frac{1}{\delta}\sum_{i=1}^{n_2} \overline{\Phi}_i(f(D_i)) \geq \frac{M - \frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i)}{M-m} f(m) + \frac{\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i) - m}{M-m} f(M). \tag{2.11}$$

On the other hand, taking into account that $m \leq \frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi}_i(B_i), \frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i) \leq M$ and using Lemma 2.1 we obtain

$$f\left(\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)\right) \leq \frac{M - \frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)}{M-m} f(m) + \frac{\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i) - m}{M-m} f(M) - \delta_f \widetilde{B} \tag{2.12}$$

and

$$f\left(\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)\right) \leq \frac{M - \frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)}{M-m} f(m) + \frac{\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i) - m}{M-m} f(M) - \delta_f \widetilde{C}, \tag{2.13}$$

where $\widetilde{B} = \frac{1}{2} - \frac{1}{M-m}\left|\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i) - \frac{m+M}{2}\right|$ and $\widetilde{C} = \frac{1}{2} - \frac{1}{M-m}\left|\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i) - \frac{m+M}{2}\right|$. Adding two inequalities (2.12) and (2.13) and putting

$$\widetilde{X} = 1 - \frac{1}{M-m}\left(\left|\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i) - \frac{m+M}{2}\right| + \left|\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i) - \frac{m+M}{2}\right|\right)$$

we obtain

$$\begin{aligned}
&f\left(\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)\right)+f\left(\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)\right)\\
&\le \frac{2M-\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)-\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)}{M-m}f(m)\\
&\quad+\frac{\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)+\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)-2m}{M-m}f(M)-\delta_f\widetilde{X}\\
&=\frac{2M-\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)-\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i)}{M-m}f(m)\\
&\quad+\frac{\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)+\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i)-2m}{M-m}f(M)-\delta_f\widetilde{X}\quad(\text{by }(2.5))\\
&\le\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(f(D_i))-\delta_f\widetilde{X},\quad(\text{by }(2.10)\text{ and }(2.11))
\end{aligned}$$

which is the first inequality in (2.6).

Furthermore, $m\le\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i),\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)\le M$. The numerical inequality $\left|t-\frac{m+M}{2}\right|\le\frac{M-m}{2}$ $(m\le t\le M)$ yields that

$$\left|\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(B_i)-\frac{m+M}{2}\right|+\left|\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)-\frac{m+M}{2}\right|\le M-m.$$

Therefore $\widetilde{X}\ge 0$. Moreover, f is convex on $[m,M]$. Hence $\delta_f\ge 0$. So the second inequality in (2.6) holds. □

Remark 2.3. We can conclude some other versions of inequality (2.6). In fact, under the assumptions in Theorem 2.2 the following inequalities hold true:

(1)
$$\begin{aligned}
\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(f(C_i))+\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(f(B_i))&\le f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right)+f\left(\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i)\right)-\delta_f\widetilde{X}_2\\
&\le f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right)+f\left(\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(D_i)\right);
\end{aligned}$$

(2)
$$\begin{aligned}
f\left(\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)\right)+\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}(f(B_i))&\le f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right)+\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(f(D_i))-\delta_f\widetilde{X}_3\\
&\le f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right)+\frac{1}{\delta}\sum_{i=1}^{n_2}\overline{\Phi}_i(f(D_i)),
\end{aligned}$$

in which

$$\widetilde{X}_2=1-\frac{1}{M-m}\left[\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i\left(\left|C_i-\frac{M+m}{2}\right|\right)+\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}\left(\left|B_i-\frac{M+m}{2}\right|\right)\right],$$

$$\widetilde{X}_3=1-\frac{1}{M-m}\left[\left|\frac{1}{\gamma}\sum_{i=1}^{n_3}\Psi_i(C_i)-\frac{M+m}{2}\right|+\frac{1}{\beta}\sum_{i=1}^{n_4}\overline{\Psi_i}\left(\left|B_i-\frac{M+m}{2}\right|\right)\right].$$

Before giving an example, we present some special cases of Theorem 2.2 which are useful in our applications. The next corollary provides a refinement of [9, Theorem 2.1].

Corollary 2.4. *Let f be a convex function on an interval J. Let $A, B, C, D \in \sigma(J)$ such that $A + D = B + C$ and $A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$. If Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$, then*

$$\begin{aligned} f(\Phi(B)) + f(\Phi(C)) &\leq \Phi(f(A)) + \Phi(f(D)) - \delta_f \widetilde{X} \\ &\leq \Phi(f(A)) + \Phi(f(D)), \end{aligned} \tag{2.14}$$

where

$$\widetilde{X} = 1 - \frac{1}{M-m}\left(\left|\Phi(B) - \frac{m+M}{2}\right| + \left|\Phi(C) - \frac{m+M}{2}\right|\right).$$

In particular,

$$f(B) + f(C) \leq f(A) + f(D) - \delta_f \widetilde{X} \leq f(A) + f(D). \tag{2.15}$$

If f is concave on J, then inequalities (2.14) and (2.15) are reversed.

Another special case of Theorem 2.2 leads to a refinement of [9, Corollary 2.3].

Corollary 2.5. *Let f be a convex function on an interval J. Let $A_i, B_i, C_i, D_i \in \sigma(J)$ $(i = 1, \cdots, n)$ such that $A_i + D_i = B_i + C_i$ and $A_i \leq m \leq B_i, C_i \leq M \leq D_i$ $(i = 1, \cdots, n)$. Let $\Phi_1, \cdots, \Phi_n$ be positive linear mappings on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$. Then*

(1) $$\begin{aligned} f\left(\sum_{i=1}^n \Phi_i(B_i)\right) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) &\leq \sum_{i=1}^n \Phi_i(f(A_i)) + \sum_{i=1}^n \Phi_i(f(D_i)) - \delta_f \widetilde{X}_1 \\ &\leq \sum_{i=1}^n \Phi_i(f(A_i)) + \sum_{i=1}^n \Phi_i(f(D_i)); \end{aligned}$$

(2) $$\begin{aligned} \sum_{i=1}^n \Phi_i(f(B_i)) + \sum_{i=1}^n \Phi_i(f(C_i)) &\leq f\left(\sum_{i=1}^n \Phi_i(A_i)\right) + f\left(\sum_{i=1}^n \Phi_i(D_i)\right) - \delta_f \widetilde{X}_2 \\ &\leq f\left(\sum_{i=1}^n \Phi_i(A_i)\right) + f\left(\sum_{i=1}^n \Phi_i(D_i)\right); \end{aligned}$$

(3) $$\begin{aligned} \sum_{i=1}^n \Phi_i(f(B_i)) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) &\leq f\left(\sum_{i=1}^n \Phi_i(D_i)\right) + \sum_{i=1}^n \Phi_i(f(A_i)) - \delta_f \widetilde{X}_3 \\ &\leq f\left(\sum_{i=1}^n \Phi_i(D_i)\right) + \sum_{i=1}^n \Phi_i(f(A_i)); \end{aligned}$$

where

$$
\begin{aligned}
\widetilde{X}_1 &= 1-\frac{1}{M-m}\left[\left|\sum_{i=1}^{n}\Phi_i(B_i)-\frac{m+M}{2}\right|+\left|\sum_{i=1}^{n}\Phi_i(C_i)-\frac{m+M}{2}\right|\right],\\
\widetilde{X}_2 &= 1-\frac{1}{M-m}\left[\sum_{i=1}^{n}\Phi_i\left(\left|B_i-\frac{m+M}{2}\right|\right)+\sum_{i=1}^{n}\Phi_i\left(\left|C_i-\frac{m+M}{2}\right|\right)\right],\\
\widetilde{X}_3 &= 1-\frac{1}{M-m}\left[\sum_{i=1}^{n}\Phi_i\left(\left|B_i-\frac{m+M}{2}\right|\right)+\left|\sum_{i=1}^{n}\Phi_i(C_i)-\frac{m+M}{2}\right|\right].
\end{aligned}
$$

Now we give an example to show that how Theorem 2.2 works.

Example 2.6. Let $n_i = 1$ for $i = 1, 2, 3, 4$ and let $f(t) = t^4$. The function f is convex but not operator convex[3]. Let $\overline{\Phi}, \Psi, \overline{\Psi} = \Phi$ in which

$$\Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C}), \quad \Phi((a_{ij})_{1\le i,j\le 3}) = (a_{ij})_{1\le i,j\le 2}.$$

If

$$A=\begin{pmatrix}1&-1&1\\-1&1&2\\1&2&-5\end{pmatrix},\ D=\begin{pmatrix}9&1&1\\1&10&2\\1&2&15\end{pmatrix},\ C=\begin{pmatrix}4&1&2\\1&4&1\\2&1&5\end{pmatrix},\ B=\begin{pmatrix}6&-1&1\\-1&7&1\\1&1&5\end{pmatrix},$$

then $\Phi(A) + \Phi(D) = \Phi(C) + \Phi(B)$ and $A \le 2.2I \le C, B \le 8I \le D$. Also $\delta_f = 2766.4$ and $\widetilde{X} = \begin{pmatrix}0.655&0.345\\0.345&0.655\end{pmatrix}$, whence

$$
(\Phi(C))^4+(\Phi(B))^4=\begin{pmatrix}1891&-859\\-859&3022\end{pmatrix}
$$

$$
\lneqq\left\{\begin{array}{l}
\begin{pmatrix}5281&2514.5\\2514.5&8758\end{pmatrix}=(\Phi(A))^4+(\Phi(D))^4-\delta_f\widetilde{X}\lneqq\begin{pmatrix}7093&3469\\3469&10570\end{pmatrix}=(\Phi(A))^4+(\Phi(D))^4\\
\begin{pmatrix}5318&2576.5\\2576.5&8867\end{pmatrix}=\Phi(A^4)+(\Phi(D))^4-\delta_f\widetilde{X}\lneqq\begin{pmatrix}7130&3531\\3531&10679\end{pmatrix}=\Phi(A^4)+(\Phi(D))^4\\
\begin{pmatrix}6202&4311.5\\4311.5&12263\end{pmatrix}=(\Phi(A))^4+\Phi(D^4)-\delta_f\widetilde{X}\lneqq\begin{pmatrix}8014&5266\\5266&14075\end{pmatrix}=(\Phi(A))^4+\Phi(D^4)\\
\begin{pmatrix}6239&4373.5\\4373.5&12372\end{pmatrix}=\Phi(A^4)+\Phi(D^4)-\delta_f\widetilde{X}\lneqq\begin{pmatrix}8051&5328\\5328&14184\end{pmatrix}=\Phi(A^4)+\Phi(D^4)
\end{array}\right.
$$

This shows that inequalities in (2.6) can be strict.

Moreover,

$$\begin{aligned}
(\Phi(A))^4 + \Phi(B^4) - \Phi(A^4) - (\Phi(B))^4 &= \begin{pmatrix} 884 & 1735 \\ 1735 & 3396 \end{pmatrix} \ngtrless 0 \\
(\Phi(A))^4 + \Phi(B^4) - \Phi(A)^4 - (\Phi(B))^4 &= \begin{pmatrix} 921 & 1797 \\ 1797 & 3505 \end{pmatrix} \ngtrless 0 \\
\Phi(A^4) + \Phi(B^4) - (\Phi(A))^4 - \Phi(B^4) &= \begin{pmatrix} 921 & 1797 \\ 1797 & 3505 \end{pmatrix} \ngtrless 0.
\end{aligned}$$

Hence there is no relationship between the right hand sides of inequalities in Corollary 2.5.

Corollary 2.7. *Let f be a convex function on an interval J. Let A_i, B_i, C_i, D_i, $i = 1, \ldots, n$, be operators in $\sigma(J)$. If $A_i \le m \le C_i, B_i \le M \le D_i$, $i = 1, \ldots, n$, for two real numbers $m < M$ and*

$$\sum_{i=1}^{n}(A_i + D_i) = \sum_{i=1}^{n}(C_i + B_i), \tag{2.16}$$

then

$$\begin{aligned}
f\left(\sum_{i=1}^{n} C_i\right) + f\left(\sum_{i=1}^{n} B_i\right) &\le f\left(\sum_{i=1}^{n} A_i\right) + f\left(\sum_{i=1}^{n} D_i\right) - \delta_{f,n}\widetilde{X}_n, \\
&\le f\left(\sum_{i=1}^{n} A_i\right) + f\left(\sum_{i=1}^{n} D_i\right)
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} f(C_i) + \sum_{i=1}^{n} f(B_i) &\le \sum_{i=1}^{n} f(A_i) + \sum_{i=1}^{n} f(D_i) - \delta_f\left(\sum_{i=1}^{n}(\widetilde{C}_i + \widetilde{B}_i)\right) \\
&\le \sum_{i=1}^{n} f(A_i) + \sum_{i=1}^{n} f(D_i)
\end{aligned} \tag{2.18}$$

in which $\delta_{f,n} = f(nm) + f(nM) - 2f\left(\frac{nm+nM}{2}\right)$ and

$$\widetilde{X}_n = 1 - \frac{1}{nM - nm}\left[\left|\sum_{i=1}^{n} C_i - \frac{nM + nm}{2}\right| + \left|\sum_{i=1}^{n} B_i - \frac{nM + nm}{2}\right|\right].$$

If f is concave, then inequalities (2.17) and (2.18) are reversed.

Proof. We prove only inequality (2.17) in the convex case. It follows from $A_i \le m \le C_i, B_i \le M \le D_i$, $(i = 1, \ldots, n)$ that

$$\sum_{i=1}^{n} A_i \le mnI \le \sum_{i=1}^{n} C_i, \ \sum_{i=1}^{n} B_i \le MnI \le \sum_{i=1}^{n} D_i.$$

Using the same reasoning as in the proof of Theorem 2.2 we get

$$
\begin{aligned}
& f\left(\sum_{i=1}^{n} C_i\right) + f\left(\sum_{i=1}^{n} B_i\right) \\
\leq\ & \frac{2Mn - \sum_{i=1}^{n}(C_i + B_i)}{Mn - mn} f(mn) + \frac{\sum_{i=1}^{n}(C_i + B_i) - 2mn}{Mn - mn} f(Mn) - \delta_{f,n}\widetilde{X}_n \\
=\ & \frac{2Mn - \sum_{i=1}^{n}(A_i + D_i)}{Mn - mn} f(mn) + \frac{\sum_{i=1}^{n}(A_i + D_i) - 2mn}{Mn - mn} f(Mn) - \delta_{f,n}\widetilde{X}_n \quad (\text{by } (2.16)) \\
\leq\ & f\left(\sum_{i=1}^{n} A_i\right) + f\left(\sum_{i=1}^{n} D_i\right) - \delta_{f,n}\widetilde{X}_n,
\end{aligned}
$$

which give the first inequality in (2.17). It is easy to see that $\delta_{f,n}\widetilde{X}_n \geq 0$, whence the second inequality derived. □

3. Applications

Using the results in Section 2, we provide some applications which are refinements of some well-known operator inequalities. As the first, we give a refinement of the operator Jensen–Mercer inequality.

Corollary 3.1. *Let $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^{n} \Phi_i(I) = I$ and $B_1, \cdots, B_n \in \sigma([m, M])$ for two scalars $m < M$. If f is a convex function on $[m, M]$, then*

$$
\begin{aligned}
f\left(m + M - \sum_{i=1}^{n} \Phi_i(B_i)\right) &\leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)) - \delta_f \widetilde{B} \\
&\leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)),
\end{aligned}
$$

where $\widetilde{B} = 1 - \dfrac{1}{M - m}\left[\sum_{i=1}^{n} \Phi_i\left(\left|B_i - \dfrac{m+M}{2}\right|\right) + \left|\sum_{i=1}^{n} \Phi_i(B_i) - \dfrac{m+M}{2}\right|\right]$.

Proof. Clearly $m \leq B_i \leq M \quad (i = 1, \cdots, n)$. Set $C_i = M + m - B_i \quad (i = 1, \cdots, n)$. Then $m \leq C_i \leq M$ and $B_i + C_i = m + M \ (i = 1, \cdots, n)$. Applying inequality (3) of Corollary 2.5 when $A_i = mI$ and $D_i = MI$ we obtain the desired inequalities. □

The next result provides a refinement of the Petrović inequality for operators.

Corollary 3.2. *If $f : [0,\infty) \to \mathbb{R}$ is a convex function and $B_1, \cdots, B_n$ are positive operators such that $\sum_{i=1}^n B_i = MI$ for some scalar $M > 0$, then*

$$\sum_{i=1}^n f(B_i) \le f\left(\sum_{i=1}^n B_i\right) + (n-1)f(0) - \delta_f \widetilde{B} \le f\left(\sum_{i=1}^n B_i\right) + (n-1)f(0),$$

where $\widetilde{B} = \frac{n}{2} - \sum_{i=1}^n \left|\frac{B_i}{M} - \frac{1}{2}\right|$.

Proof. It follows from $0 \le B_i \le M$ that

$$f(B_i) \le \frac{M - B_i}{M - 0} f(0) + \frac{B_i - 0}{M - 0} f(M) - \delta_f \widetilde{B_i} \quad (i = 1, \cdots, n).$$

Summing above inequalities over i we get

$$\begin{aligned}\sum_{i=1}^n f(B_i) &\le \frac{nM - \sum_{i=1}^n B_i}{M} f(0) + \frac{\sum_{i=1}^n B_i}{M} f(M) - \delta_f \sum_{i=1}^n \widetilde{B_i} \\ &= (n-1)f(0) + f\left(\sum_{i=1}^n B_i\right) - \delta_f \widetilde{B} \quad (\text{by } \textstyle\sum_{i=1}^n B_i = M) \\ &\le (n-1)f(0) + f\left(\sum_{i=1}^n B_i\right) \quad (\text{by } \delta_f \widetilde{B} \ge 0).\end{aligned}$$

where $\widetilde{B} = \frac{n}{2} - \sum_{i=1}^n \left|\frac{B_i}{M} - \frac{1}{2}\right|$. □

As another consequence of Theorem 2.2, we present a refinement of the Jensen operator inequality for real convex functions. The authors of [9] introduce a subset Ω of $\mathbb{B}_h(\mathscr{H}) \times \mathbb{B}_h(\mathscr{H})$ defined by

$$\Omega = \left\{(A, B) \;\middle|\; A \le m \le \frac{A+B}{2} \le M \le B, \quad \text{for some} \quad m, M \in \mathbb{R}\right\}.$$

We have the following result.

Corollary 3.3. *Let f be a convex function on an interval J containing m, M. Let Φ_i, $i = 1, \ldots, n$, be positive linear mappings on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$. If $(A_i, D_i) \in \Omega$, $i = 1, \ldots, n$, then*

$$\begin{aligned} f\left(\sum_{i=1}^n \Phi_i\left(\frac{A_i + D_i}{2}\right)\right) &\le \sum_{i=1}^n \Phi_i\left(\frac{f(A_i) + f(D_i)}{2}\right) - \delta_f \widetilde{X} \\ &\le \sum_{i=1}^n \Phi_i\left(\frac{f(A_i) + f(D_i)}{2}\right), \end{aligned} \tag{3.1}$$

where

$$\widetilde{X} = \frac{1}{2} - \frac{1}{M-m}\left|\sum_{i=1}^{n}\Phi_i\left(\frac{A_i+D_i}{2}\right) - \frac{m+M}{2}\right|.$$

If f is concave, then inequalities in (3.1) are reversed.

Proof. Putting $B_i = C_i = \frac{A_i+D_i}{2}$ and using inequality (1) of Corollary 2.5, we conclude the desired result. □

Note that utilizing Corollary 2.5, we even be able to obtain a converse of the Jensen operator inequality. For this end, under the assumptions in the Corollary 3.3 we have

$$\begin{aligned}\sum_{i=1}^{n}\Phi_i\left(f\left(\frac{A_i+D_i}{2}\right)\right) &\le \frac{1}{2}\left[f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) + f\left(\sum_{i=1}^{n}\Phi_i(D_i)\right)\right] - \delta_f\widetilde{X} \\ &\le \frac{1}{2}\left[f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) + f\left(\sum_{i=1}^{n}\Phi_i(D_i)\right)\right], \end{aligned} \tag{3.2}$$

where

$$\widetilde{X} = \frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\Phi_i\left(\left|\frac{A_i+D_i}{2} - \frac{m+M}{2}\right|\right).$$

Note that the function f need not to be operator convex. Let us give an example to illustrate these inequalities.

Example 3.4. Let $n = 1$ and the unital positive linear map $\Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ be defined by

$$\Phi((a_{ij})_{1\le i,j\le 3}) = (a_{ij})_{1\le i,j\le 2}$$

for each $A = (a_{ij})_{1\le i,j\le 3} \in \mathcal{M}_3(\mathbb{C})$. Consider the convex function $f(t) = e^t$ on $[0,\infty)$. If

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 7 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix},$$

then $0 \le A \le 2I \le \frac{A+D}{2} \le 5I \le D$, i.e., $(A, D) \in \Omega$. Hence it follows from (3.1) that

$$\begin{aligned} f\left(\Phi\left(\frac{A+D}{2}\right)\right) = \begin{pmatrix} 79.8 & -50.5 \\ -50.5 & 54.6 \end{pmatrix} &\lneqq \begin{pmatrix} 759.2 & -399 \\ -399 & 344 \end{pmatrix} = \Phi\left(\frac{f(A)+f(D)}{2}\right) - \delta_f\widetilde{X} \\ &\lneqq \begin{pmatrix} 768.2 & -408 \\ -408 & 362 \end{pmatrix} = \Phi\left(\frac{f(A)+f(D)}{2}\right), \end{aligned}$$

in which $\delta_f = 89.6$ and $\widetilde{X} = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}$.

It should be mentioned that in the case when f is operator convex, under the assumptions in Corollary 3.3 we have even more:

$$\begin{aligned} f\left(\sum_{i=1}^n \Phi_i\left(\frac{A_i+D_i}{2}\right)\right) &\le \sum_{i=1}^n \Phi_i\left(f\left(\frac{A_i+D_i}{2}\right)\right) \quad \text{(by the Jensen inequality)} \\ &\le \frac{1}{2}\left[f\left(\sum_{i=1}^n \Phi_i(A_i)\right) + f\left(\sum_{i=1}^n \Phi_i(D_i)\right)\right] - \delta_f \widetilde{X} \quad \text{(by (3.2))} \\ &\le \frac{1}{2}\left[\sum_{i=1}^n \Phi_i(f(A_i)+f(D_i))\right] - \delta_f \widetilde{X} \quad \text{(by the Jensen inequality)} \\ &\le \sum_{i=1}^n \Phi_i\left(\frac{f(A_i)+f(D_i)}{2}\right) \quad \text{(since } \delta_f \widetilde{X} \ge 0). \end{aligned}$$

Corollary 3.5. *If f is a convex function on an interval J containing m, M, then*

$$\begin{aligned} f(\lambda A + (1-\lambda)D) &\le \lambda f(A) + (1-\lambda) f(D) - \delta_f \widetilde{X} \\ &\le \lambda f(A) + (1-\lambda) f(D) \end{aligned} \tag{3.3}$$

for all $(A, D) \in \Omega$ and all $\lambda \in [0,1]$, where $\widetilde{X} = \frac{1}{2} - \frac{1}{M-m}\left|\frac{A+D-M-m}{2}\right|$. If f is concave, then inequality (3.3) is reversed.

Proof. Put $n = 1$ and let Φ be the identity map in Corollary 3.3 to get

$$f\left(\frac{A+D}{2}\right) \le \frac{f(A)+f(D)}{2} - \delta_f \widetilde{X} \le \frac{f(A)+f(D)}{2}$$

for any $(A, D) \in \Omega$, which implies (3.3) by the continuity of f. $\square$

Regarding to obtain an operator version of (3.4), it is shown in [9] that if $f : [0,\infty) \to [0,\infty)$ is a convex function with $f(0) \le 0$, then

$$f(A) + f(B) \le f(A+B) \tag{3.4}$$

for all strictly positive operators A, B for which $A \le M \le A+B$ and $B \le M \le A+B$ for some scalar M. We give a refined extension of this result as follows.

Theorem 3.6. *If $f : [0,\infty) \to \mathbb{R}$ is a convex function with $f(0) \le 0$ then*

$$\sum_{i=1}^n f(C_i) \le f\left(\sum_{i=1}^n C_i\right) - \delta_f \sum_{i=1}^n \widetilde{C}_i \le f\left(\sum_{i=1}^n C_i\right) \tag{3.5}$$

for all positive operators C_i such that $C_i \le M \le \sum_{i=1}^n C_i$ $(i = 1, \ldots, n)$ for some scalar $M \ge 0$. If f is concave, then the reverse inequality is valid in (3.5).

In particular, if f is convex, then

$$f(A) + f(B) \le f(A + B) - \delta_f \widetilde{X} \le f(A + B)$$

for all positive operators A, B such that $A \le MI \le A + B$ and $B \le MI \le A + B$ for some scalar $M \ge 0$, where $\widetilde{X} = 1 - \left|\frac{A}{M} - \frac{1}{2}\right| - \left|\frac{B}{M} - \frac{1}{2}\right|$.

Proof. Without loss of generality let $M > 0$. Lemma 2.1 implies that

$$f(C_i) \le \frac{MI - C_i}{M - 0} f(0) + \frac{C_i}{M - 0} f(M) - \delta_f \widetilde{C}_i = \frac{C_i}{M} f(M) - \delta_f \widetilde{C}_i \quad (i = 1, \ldots, n)$$

since $f(0) \le 0$. Summing the above inequalities over i we get

$$\sum_{i=1}^n f(C_i) \le \frac{\sum_{i=1}^n C_i}{M} f(M) - \delta_f \sum_{i=1}^n \widetilde{C}_i. \tag{3.6}$$

Since the spectrum of $\sum_{i=1}^n C_i$ is contained in $[M, \infty) \subset [0, \infty) \setminus [0, M)$, we have

$$\begin{aligned} f\left(\sum_{i=1}^n C_i\right) &\ge \frac{MI - \sum_{i=1}^n C_i}{M - 0} f(0) + \frac{\sum_{i=1}^n C_i}{M - 0} f(M) \\ &\ge \frac{\sum_{i=1}^n C_i}{M} f(M) \quad (\text{since } MI \le \textstyle\sum_{i=1}^n C_i \text{ and } f(0) \le 0). \end{aligned} \tag{3.7}$$

Combining two inequalities (3.6) and (3.7), we reach to the desired inequality (3.5). □

Theorem 3.7. *Let $A, B, C, D \in \sigma(J)$ such that $A \le m \le B, C \le M \le D$ for two real numbers $m < M$. If f is a convex function on J and any one of the following conditions*

$$\text{(i)} \quad B + C \le A + D \quad \textit{and} \quad f(m) \le f(M)$$

$$\text{(ii)} \quad A + D \le B + C \quad \textit{and} \quad f(M) \le f(m)$$

is satisfied, then

$$f(B) + f(C) \le f(A) + f(D) - \delta_f \widetilde{X} \le f(A) + f(D), \tag{3.8}$$

where $\widetilde{X} = 1 - \frac{1}{M - m}\left(\left|B - \frac{M + m}{2}\right| + \left|C - \frac{M + m}{2}\right|\right)$.

If f is concave and any one of the following conditions

$$\text{(iii)} \quad B + C \le A + D \quad \textit{and} \quad f(M) \le f(m)$$

$$\text{(iv)} \quad A + D \le B + C \quad \textit{and} \quad f(m) \le f(M)$$

is satisfied, then inequality (3.8) is reversed.

Proof. Let f be convex and (i) is valid. It follows from Lemma 2.1 that

$$f(B) \leq \frac{f(M)-f(m)}{M-m}B + \frac{f(m)M-f(M)m}{M-m} - \delta_f\left(\frac{1}{2} - \frac{1}{M-m}\left|B - \frac{M+m}{2}\right|\right)$$

and

$$f(C) \leq \frac{f(M)-f(m)}{M-m}C + \frac{f(m)M-f(M)m}{M-m} - \delta_f\left(\frac{1}{2} - \frac{1}{M-m}\left|C - \frac{M+m}{2}\right|\right).$$

Summing above inequalities we get

$$\begin{aligned}
f(B)+f(C) &\leq \frac{f(M)-f(m)}{M-m}(B+C) + 2\,\frac{f(m)M-f(M)m}{M-m} - \delta_f\widetilde{X} \\
&\leq \frac{f(M)-f(m)}{M-m}(A+D) + 2\,\frac{f(m)M-f(M)m}{M-m} - \delta_f\widetilde{X} \quad \text{(by (i))} \\
&= \frac{f(M)-f(m)}{M-m}A + \frac{f(m)M-f(M)m}{M-m} \\
&\quad + \frac{f(M)-f(m)}{M-m}D + \frac{f(m)M-f(M)m}{M-m} - \delta_f\widetilde{X} \\
&\leq f(A)+f(D) - \delta_f\widetilde{X} \qquad \text{(by (2.8) and (2.9))} \\
&\leq f(A)+f(D) \qquad (\text{by } \delta_f\widetilde{X} \geq 0\ )
\end{aligned}$$

The other cases can be verified similarly. □

Applying the above theorem to the power functions we get

Corollary 3.8. *Let $A, B, C, D \in \mathbb{B}_h(\mathscr{H})$ be such that $I \leq A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$. If one of the following conditions*

$$\text{(i)}\quad B+C \leq A+D \quad \textit{and} \quad p \geq 1$$

$$\text{(ii)}\quad A+D \leq B+C \quad \textit{and} \quad p \leq 0$$

is satisfied, then

$$B^p + C^p \leq A^q + D^q - \delta_p\widetilde{X} \leq A^q + D^q$$

for each $q \geq p$, where

$$\delta_p = m^p + M^p - 2\left(\frac{m+M}{2}\right)^p, \quad \widetilde{X} = 1 - \frac{1}{M-m}\left(\left|B - \frac{M+m}{2}\right| + \left|C - \frac{M+m}{2}\right|\right).$$

Proof. Let (i) be valid. Applying Theorem 3.7 for $f(t) = t^p$, it follows

$$\begin{aligned}
B^p + C^p &\leq A^p + D^p - \delta_p\widetilde{X} \\
&\leq A^q + D^q - \delta_p\widetilde{X} \qquad (\text{by } q \geq p) \\
&\leq A^q + D^q \qquad (\text{by } \delta_p\widetilde{X} \geq 0\ )
\end{aligned}$$

The other cases may be verified similarly. □

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